The Term Structure of Government Debt Uncertainty

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Uncertainty and volatility in financial markets

"The Fed had made the life of money market operators quite easy by taking volatility out of the marketplace. When you move in one direction, it is pretty easy for people to assume that that direction will continue forever. Maintaining and suppressing rates for a prolonged period has generated an environment of complacency, and it's a natural instinct to drive people to take greater risk. As long as it's not extreme or destabilizing, more volatility in the market is healthy. I personally have no qualms about injecting a bit more volatility into the marketplace particularly now when we've bought time away from the most severe crisis imaginable, certainly since the Great Depression."

Richard Fisher, Dallas Fed president
March 24, 2014 (London School of Economics)

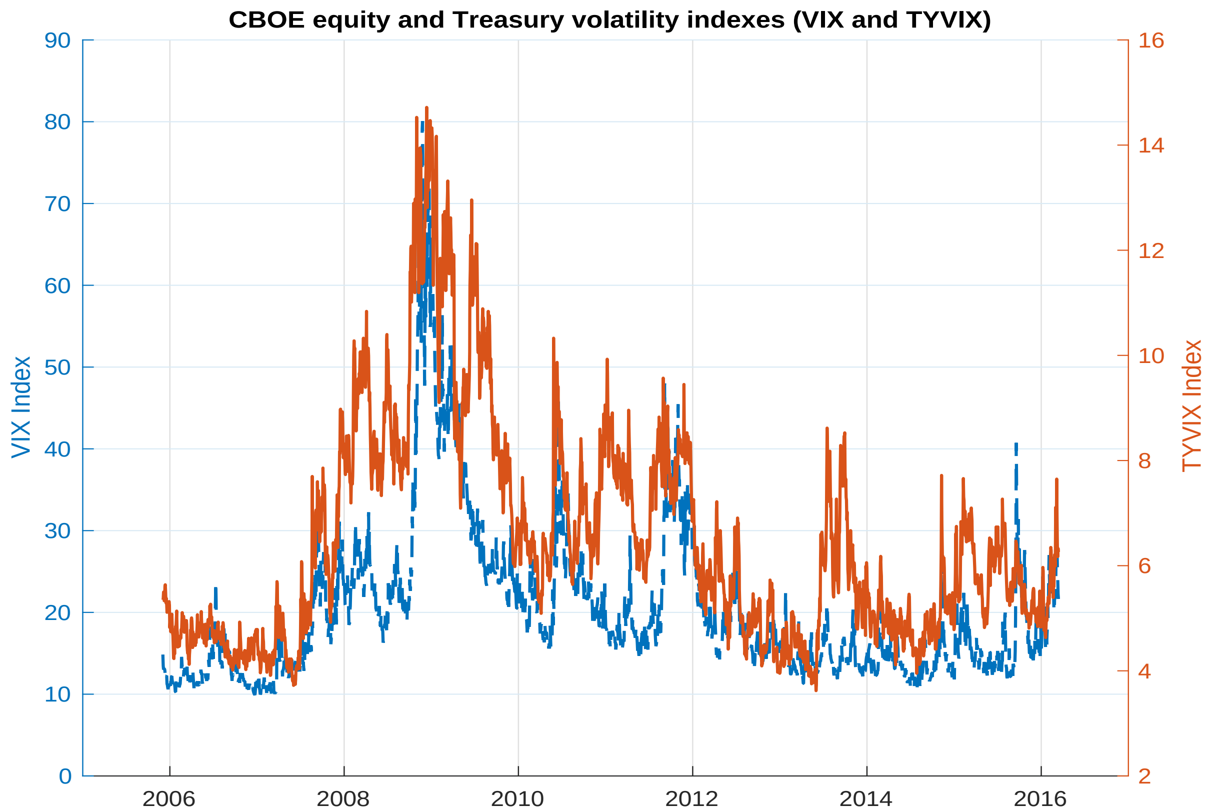
Purpose: Government debt uncertainty

- How do markets price uncertainty around interest rate movements?
- Asset volatility: most of the insights have been made in the equity space
- How does benchmarking work in the fixed income volatility space?
- How is asset evaluation affected in these markets by or in anticipation of tail events such as the Lehman's collapse or Fed rate hikes?
- What type of derivative instruments would be able to mitigate losses arising from these events?
- Paper addresses these questions in relation to government bond markets

Government bond markets

- Consider a model with random government bond volatility and evaluate derivatives written on expected volatility (futures and options)
- These derivatives provide valuable information regarding the term structure of expectations and the risk premiums required to invest in these markets
  - Well beyond information contained by variance swap strikes (reviewed below)
- Recent financial history provides us with an interesting laboratory where to test the predictions of our model
- Cboe TYVIX = Treasury VIX

Equity and interest rate volatility: tête-à-tête



A contingent claim model of expected gvt bond volatility

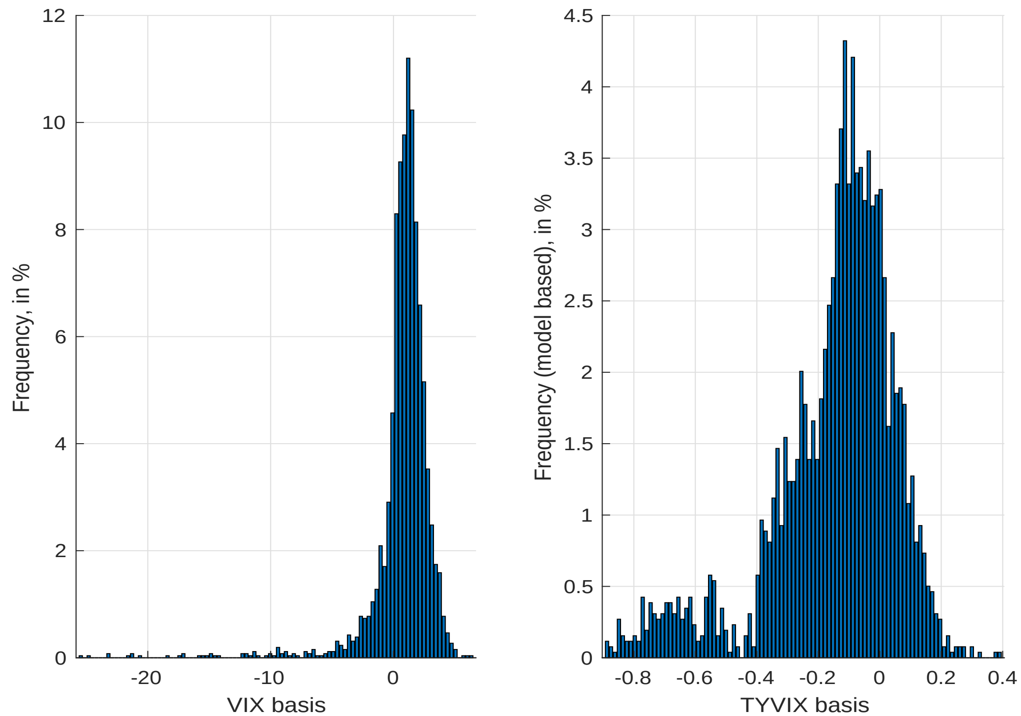
- Construct model to evaluate derivatives on variance swap rates such as TYVIX, in addition to TYVIX

- Scope is practically relevant
  - Indices of expected volatility such as VIX or TYVIX are not investable
  - VIX futures and options are traded
  - TYVIX futures have been introduced already while options have not yet
  - Previous figure suggests that fixed income vol has content beyond equity

- Paper introduces the first model to price derivatives on expected government bond volatility
  - Derivatives on fixed income expected vol might well exhibit different behavior than those on VIX

Our conclusions in a nutshell

Contango, backwardation



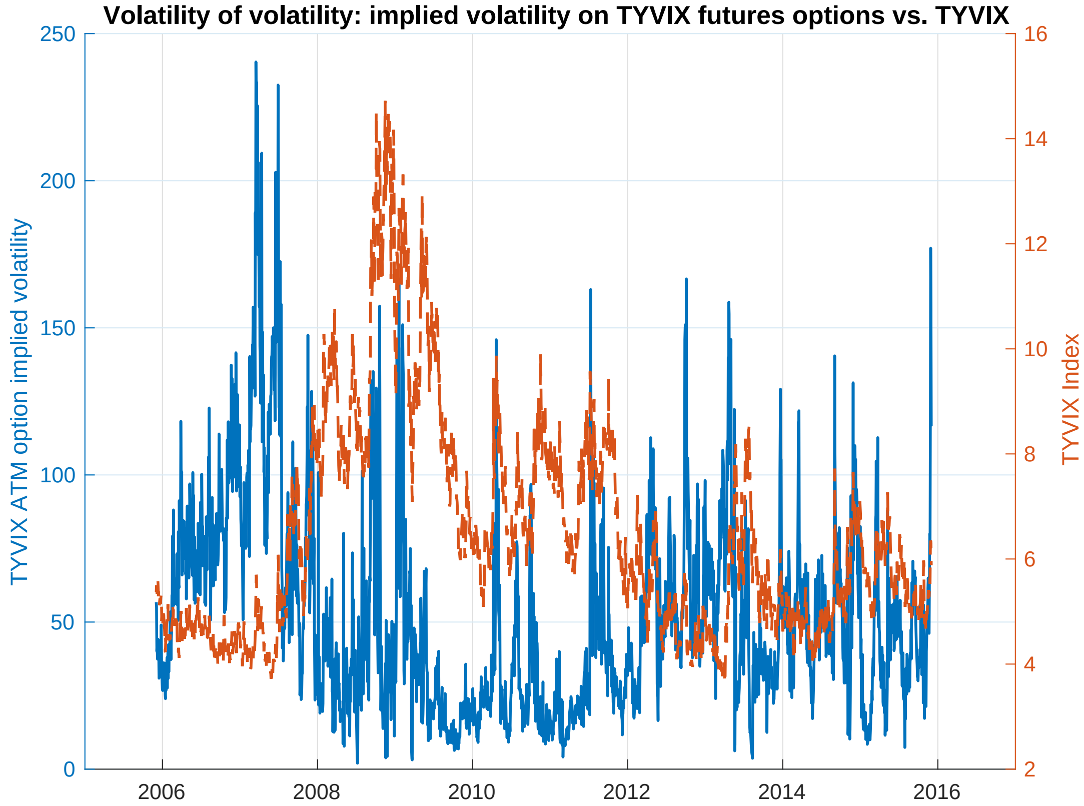
Properties and mechanism

- VIX futures curve is often in contango
  - perhaps reflecting risk aversion of the bid side of the market for volatility
- Our calibrated model suggests markets for gvt bond volatility work quite differently
  - TYVIX futures markets may be more often in backwardation — frequently come with an inverted future curve
  - Not an arbitrage opportunity, but it is puzzling

- Model
  - Interest rate uncertainty is driven by a short-term and a long-term factor
  - When uncertainty increases, the term structure of gvt volatility futures shifts up
    - the long-term uncertainty factor mean-reverts at a very fast pace
    - curve enters and remains in backwardation for a while

- Pricing of volatility of volatility (options on futures on TYVIX)
  - Not surprisingly vol of vol displays anticipatory power regarding tail events

Volatility of volatility



Implications

- Hedging
- Risk management
- Information content regarding monetary policy
- Macroprudential analysis

Related literature

- Contingent claim theory for fixed income vol much more recent than equity

- Mele and Obayashi (2015): expected volatility in a variety of segments in the fixed income universe
  - interest rate swaps
  - government bonds
  - time-deposits
  - credit

- They identify numéraire and market probability to evaluate variance swaps in each of these markets

- This paper considers derivatives on such variance swap strikes in the gvt bond market

- Cieślak and Povala (2016) also model the term structure of expected volatility of yields (a model distinct from ours)
  - do not consider the pricing of derivatives written on this volatility

- Choi, Mueller and Vedolin (2017) consider variance swaps in Treasury markets and price them consistent with Mele and Obayashi (2015)
  - Pricing of variance swaps is one of the starting points of our analysis
  - Futures and options on TYVIX provide information that goes well beyond TYVIX
    - ∗ Selling realized vol against a fixed strike may be profitable
    - ∗ But selling futures on expected volatility may be far from it, due to backwardation
  - Furthermore, we investigate higher order properties of expected volatility –
    - ∗ volatility of volatility

- Trolle and Schwartz (2014): interest rate swap (IRS)
  - Consider empirical studies of variance risk premiums in the IRS market and, following Mele and Obayashi (2015), price an IRS variance swap by re-scaling the variance payoff by the annuity factor
- Additional literature in the review section of the paper

- Three features

  1. Accounts for the endogenous nature of government bond vol—fixed income vol needs satisfy no-arbitrage restrictions jointly with the underlying bond price
     * Equity works much more simply
  2. Variance swaps and derivatives upon them can be priced and (statically) hedged whilst ensuring the entire yield curve is fitted without error
     * Appealing for reasons related to market making
     * Appealing as it allows the whole yield curve to feed information about the direction of future volatility
  3. Quasi-closed form solutions that can be readily implemented despite the high-dimensional no-arbitrage restrictions that underlie the model dynamics

Outline of the rest of the presentation

1. Assets underlying government bond volatility

2. A model of forward bond price volatility

3. Government bond volatility derivatives

4. Model predictions

1. Assets underlying gvt bond volatility

Forward prices

Coupon bearing bond

$$P_t^c(\mathbb{T}) \equiv \sum_{i=i_t}^{N} \frac{C_i}{n} P_t(T_i) + P_t(\mathbb{T})$$

Forward coupon bearing bond

$$F_t(S, \mathbb{T}) = \frac{P_t^c(\mathbb{T})}{P_t(S)}$$

Martingale under the forward prob

$$\frac{dF_\tau(S, \mathbb{T})}{F_\tau(S, \mathbb{T})} = v_\tau(S, \mathbb{T}) \cdot dW_\tau^S, \quad \tau \in (t, S)$$

Government bond variance swap

Payoff

$$\pi\left(T,\mathbb{T}\right)\equiv\int_t^T\left\|v_\tau\left(S,\mathbb{T}\right)\right\|^2d\tau-\mathbb{P}\left(t,T,S,\mathbb{T}\right),\quad T\leq S$$

Strike

$$\mathbb{P}\left(t,T,S,\mathbb{T}\right)=\frac{1}{P_t\left(T\right)}\mathbb{E}_t\left[e^{-\int_t^T r_\tau d\tau}\int_t^T\left\|v_\tau\left(S,\mathbb{T}\right)\right\|^2d\tau\right]=\mathbb{E}_t^{Q_T}\left(V_t\left(T,S,\mathbb{T}\right)\right)$$

Duration mismatch

$$\frac{dF_\tau\left(S,\mathbb{T}\right)}{F_\tau\left(S,\mathbb{T}\right)}=\underbrace{v_\tau\left(S,\mathbb{T}\right)\left(v_\tau\left(S,\mathbb{T}\right)-v_\tau\left(T,\mathbb{T}\right)\right)}_{\equiv z_\tau(T,S,\mathbb{T})}d\tau+v_\tau\left(S,\mathbb{T}\right)\cdot dW_\tau^T,\quad\tau\in\left(t,T\right)$$

Proposition 1

$$\mathbb{P}\left(t,T,S,\mathbb{T}\right)=2\left(1-\mathbb{E}_t^{Q_T}\left(e^{\bar{z}(\tau,T,S,\mathbb{T})}-\bar{z}\left(t,T,S,\mathbb{T}\right)\right)\right)$$
$$+\frac{2}{P_t\left(T\right)}\left(\int_0^{F_t(S,\mathbb{T})}\mathrm{Put}_t\left(K\right)\frac{1}{K^2}dK+\int_{F_t(S,\mathbb{T})}^{\infty}\mathrm{Call}_t\left(K\right)\frac{1}{K^2}dK\right)$$

where $\mathrm{Put}_t\left(K\right)$ *and* $\mathrm{Call}_t\left(K\right)$ *denote the prices of European puts and calls, written on bond forwards, struck at* K*, and* $\bar{z}\left(t,T,S,\mathbb{T}\right)\equiv\int_t^T z_\tau\left(T,S,\mathbb{T}\right)d\tau$

Index ⟶

$$\boxed{\mathrm{GB\text{-}VI}(t,T,S,\mathbb{T})\equiv\sqrt{\frac{1}{T-t}\mathbb{P}\left(t,T,S,\mathbb{T}\right)}}$$

In practice

- $\bar{z} = 0$
- American options
- Cboe TYVIX

2. A model of forward bond price volatility

Fluctuating uncertainties and spot prices

Short-term rate r_τ is mean-reverting with mean-reverting stochastic volatility; *expected* short-term basis point variance is mean-reverting, and driven by an independent factor

$$\begin{cases} dr_\tau &= \kappa_r \left(\theta_\tau - r_\tau\right) d\tau + v_\tau dW_{1\tau} \\ dv_\tau^2 &= \kappa_v \left(\mu_\tau - v_\tau^2\right) d\tau + \xi v_\tau \left(\rho dW_{1\tau} + \sqrt{1-\rho^2} dW_{2\tau}\right) \\ d\mu_\tau &= \kappa_\mu \left(m - \mu_\tau\right) d\tau + \gamma \sqrt{\mu_\tau} dW_{3\tau} \end{cases}$$

θ_τ: infinite dimensional parameter
chosen to calibrate the yield curve without errors

- First, we provide preliminary background to evaluate the whole yield curve we reference upon in this paper

- **Proposition 2**. *Price of a zero coupon bond expiring at* S *when the state is* $(r_\tau, v_\tau, \mu_\tau)$, *is*

$$P_\tau\left(r_\tau, v_\tau^2, \mu_t, S\right) \equiv e^{A_S(\tau)-B_S(\tau)r_\tau+C_S(\tau)v_\tau^2+D_S(\tau)\mu_\tau},$$

for three functions $A_S(\cdot)$, $B_S(\cdot)$ *and* $C_S(\cdot)$

- **Forward bond price**

$$F_t(S,\mathbb{T}) \equiv F_t\left(r_t, v_t^2, \mu_t, S, \mathbb{T}\right) \equiv \sum_{i=i_t}^{N} \frac{C_i}{n} F_t^z\left(r_t, v_t^2, \mu_t, S, T_i\right) + F_t^z\left(r_t, v_t^2, \mu_t, S, \mathbb{T}\right),$$

where

$$F_t^z(S, T_o) \equiv F_t^z\left(r_t, v_t^2, \mu_t, S, T_o\right) \equiv \frac{P_t\left(r_t, v_t^2, \mu_t, T_o\right)}{P_t\left(r_t, v_t^2, \mu_t, S\right)}$$

Bond price volatility

Zero coupon bonds—forwards

$$\begin{cases} \frac{dF_{\tau}^{z}(S,T_O)}{F_{\tau}^{z}(S,T_O)} & = & (\cdots)\,d\tau + v_{\tau}\left(v_{1\tau}^{z}(S,T_O)\,dW_{1\tau}^{T} + v_{2\tau}^{z}(S,T_O)\,dW_{2\tau}^{T}\right) + \sqrt{\mu_{\tau}}v_{3\tau}^{z}(S,T_O)\,dW_{3\tau}^{T} \\ dv_{\tau}^{2} & = & \left(\kappa_{v}\mu_{\tau} - G_{T}(\tau)\,v_{\tau}^{2}\right)d\tau + \xi v_{\tau}\left(\rho dW_{1\tau}^{T} + \sqrt{1-\rho^{2}}dW_{2\tau}^{T}\right) \\ d\mu_{\tau} & = & \left(\kappa_{\mu}m - H_{T}(\tau)\,\mu_{\tau}\right)d\tau + \gamma\sqrt{\mu_{\tau}}dW_{3\tau}^{T} \end{cases}$$

for three volatility coefficients $v_{i\tau}^{z}(S,T_o) = \varphi_{i\tau}(T_o) - \varphi_{i\tau}(S)\,, \quad i = 1,2,3$
(and some deterministic functions G_T and H_T)

Zero forward volatility

$$v_{\tau}^{z}(v_{\tau},\mu_{t};S,T_o) \equiv [v_{\tau}^{z}(v_{\tau};S,T_o) \quad \sqrt{\mu_{\tau}}v_{3\tau}^{z}(S,T_o)],$$

where $v_{\tau}^{z}(v_{\tau};S,T_o) = v_{\tau} \times [v_{1\tau}^{z}(S,T_o) \quad v_{2\tau}^{z}(S,T_o)]$

Proposition 3. *Realized instantaneous vol is*

$$\left\| v_{\tau}^{z}\left(v_{\tau}, \mu_{t}; S, T_{o}\right)\right\|^{2} = \phi_{1\tau}^{z}\left(S, T_{o}\right) v_{\tau}^{2} + \phi_{2\tau}^{z}\left(S, T_{o}\right) \mu_{\tau}$$

and the index of bond price volatility is

$$\begin{aligned} & \text{GB-VI}^{z}\left(v_{t}^{2}, \mu_{t}; t, T, S, T_{o}\right) \\ & \equiv \sqrt{\kappa_{m} m \mathcal{B}_{1}^{z}\left(t, T, S, T_{o}\right) + \mathcal{B}_{2}^{z}\left(t, T, S, T_{o}\right) v_{t}^{2} + \mathcal{B}_{3}^{z}\left(t, T, S, T_{o}\right) \mu_{t}} \end{aligned}$$

where $\phi_{j\tau}^{z}\left(S, T_{o}\right)$ *are two deterministic functions of calendar time* τ*, and* $\mathcal{B}_{j}^{z}\left(t, T, S, T_{o}\right)$ *are three constants*

“Long-run risks”

Coupon bearing bonds and volatility feedbacks

Realized volatility

$$\left\|v_{\tau}\left(v_{\tau},\mu_{\tau};S,\mathbb{T}\right)\right\|^{2}$$
$$\equiv\left(\sum_{k=1}^{2}\left(\sum_{j=i_t}^{N}\omega_{\tau}^{j}\left(S,\mathbb{T}\right)v_{k\tau}^{z}\left(S,T_j\right)\right)^{2}\right)v_{\tau}^{2}+\left(\sum_{j=i_t}^{N}\omega_{\tau}^{j}\left(S,\mathbb{T}\right)v_{3\tau}^{z}\left(S,T_j\right)\right)^{2}\mu_{\tau}$$

where $v_{k\tau}^{z}\left(S,T_j\right)$ are the zero exposures, and

$$\omega_{\tau}^{i}\left(S,\mathbb{T}\right)\equiv\bar{C}_i\frac{F_{\tau}^{z}\left(S,T_i\right)}{F_{\tau}\left(S,\mathbb{T}\right)}$$

Need approximations

Rebonato's type of tricks: freeze $\omega_{\tau}^{i}\left(S,\mathbb{T}\right)$ at $\omega_{t}^{i}\left(S,\mathbb{T}\right)$

Proposition 4 *The index of bond price volatility is*

$$\begin{aligned} & \text{GB-VI}(\mathsf{Y}^{\$}_{t,\mathbb{T}}, v_t^2, \mu_t; t, T, S, \mathbb{T}) \\ = \quad & \sqrt{\kappa_\mu m \mathcal{B}_1(\mathsf{Y}^{\$}_{t,\mathbb{T}}, T, S, \mathbb{T}) + \mathcal{B}_2(\mathsf{Y}^{\$}_{t,\mathbb{T}}, T, S, \mathbb{T}) v_t^2 + \mathcal{B}_3(\mathsf{Y}^{\$}_{t,\mathbb{T}}, T, S, \mathbb{T}) \mu_t}, \end{aligned}$$

where $\mathcal{B}_i(Y^{\$}_{t,\mathbb{T}}, \cdot, \cdot, \cdot)$ *are three functions and* $\mathsf{Y}^{\$}_{t,\mathbb{T}} \equiv \{y_t^{\$}(\tau)\}_{\tau \in (0, \mathbb{T}-t)}$ *is the entire yield curve observed at* t *and up to time to maturity* $T - t$*.*

3. Government bond volatility derivatives

Futures on TYVIX

Already listed

Value at t of a TYVIX future maturing at $t+\delta$, approximated through

$$\begin{aligned}&\hat{\mathcal{F}}(\mathsf{Y}^{\$}_{t,\mathbb{T}}, v_t^2, \mu_t; t+\delta, T, S, \mathbb{T})\\ &\equiv \iint \text{GB-VI}(\mathsf{Y}^{\$}_{t,\mathbb{T}}, x, y; t, T, S, \mathbb{T}) \cdot f_\delta \left(x, y \right| v_t^2, \mu_t \big) \, dx dy,\end{aligned}$$

where the transition density $f_\delta \left(x, y \right| v_t^2, \mu_t \big)$ is known in closed form

Options

Not listed yet

Options on TYVIX

Value at t of a TYVIX European call option maturing at $t+\Delta$ and struck at K, approximated through

$$\hat{\mathcal{C}}(\mathsf{Y}_{t,\mathbb{T}}^{\$}, v_t^2, \mu_t; t+\Delta, K, T, S, \mathbb{T})$$
$$\equiv P_t(T) \iint \left(\text{GB-VI}(\mathsf{Y}_{t,\mathbb{T}}^{\$}, x, y; t, T, S, \mathbb{T}) - K\right)^+ f_\Delta^F\left(x, y | v_t^2, \mu_t\right) dxdy,$$

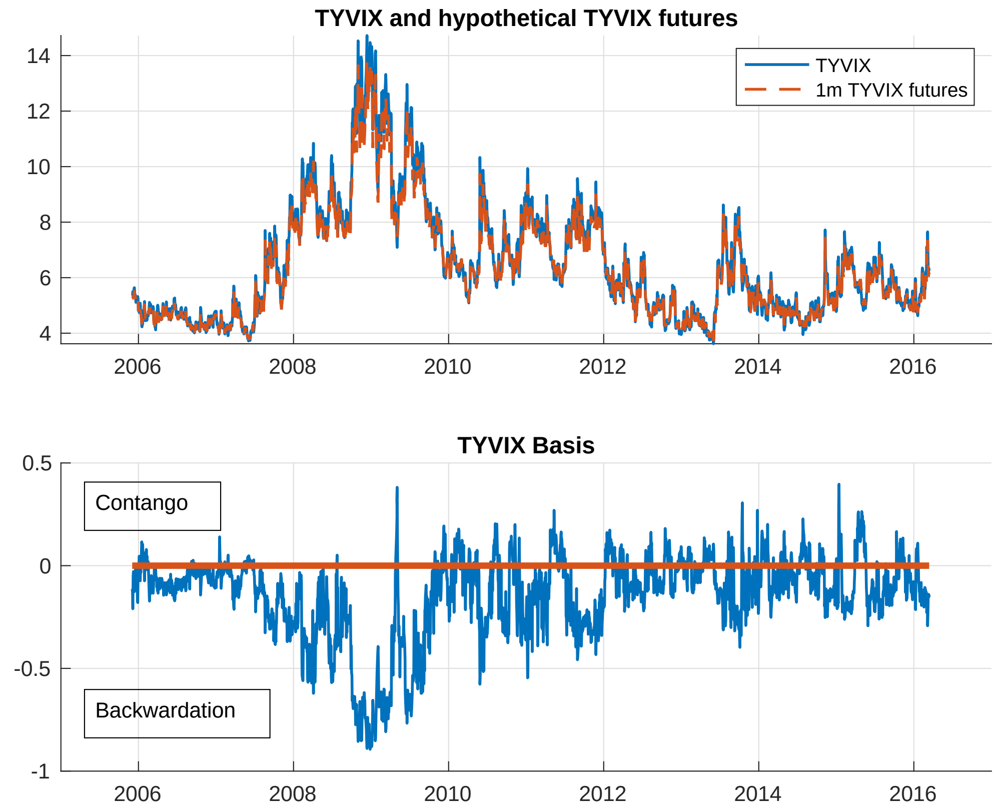
where the transition density $f_\Delta^F\left(x, y | v_t^2, \mu_t\right)$ is known in closed form

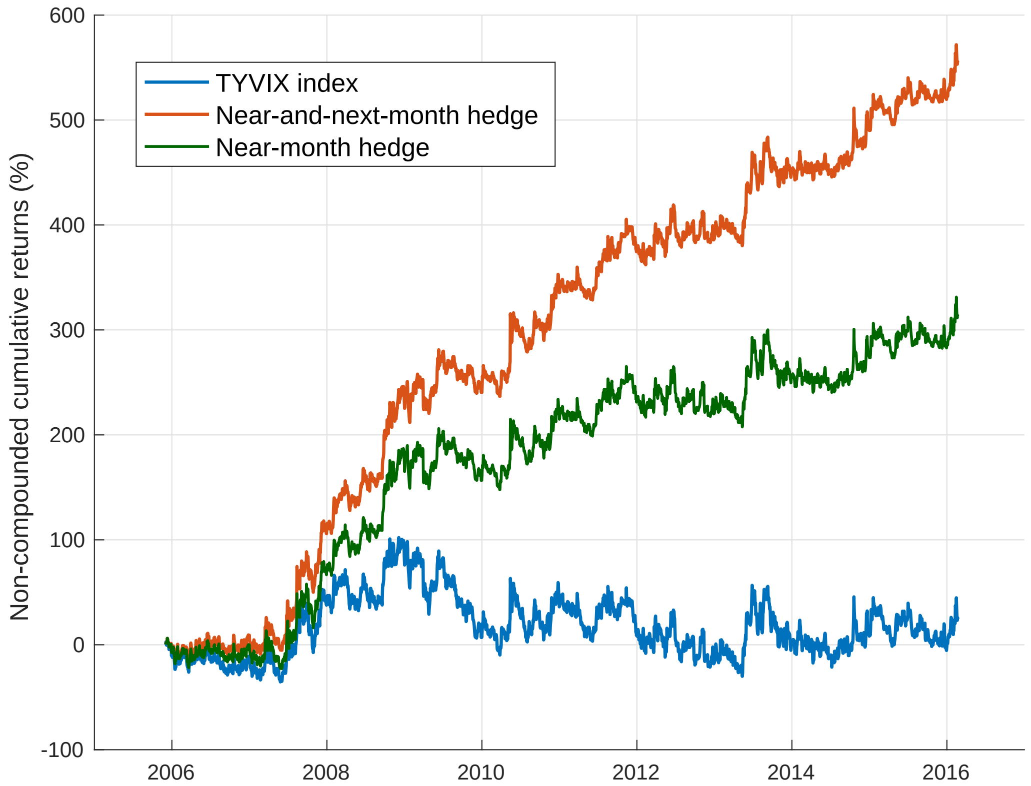
Options on futures on TYVIX

More likely to be listed

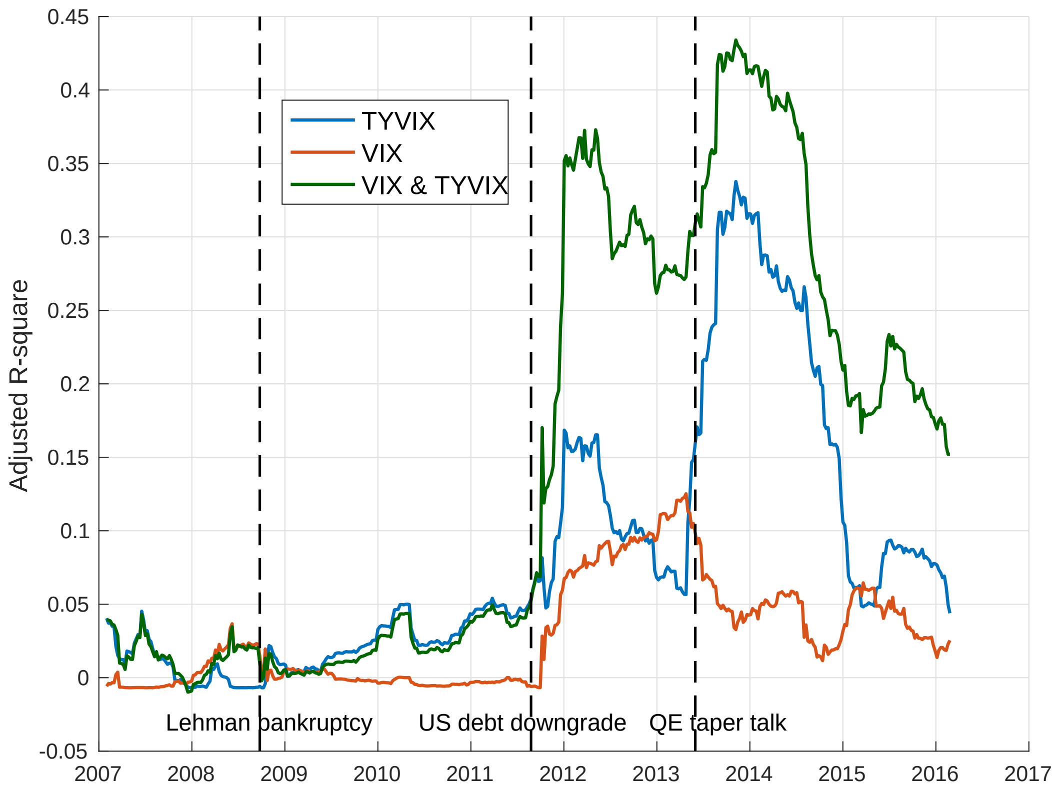
4. Model predictions

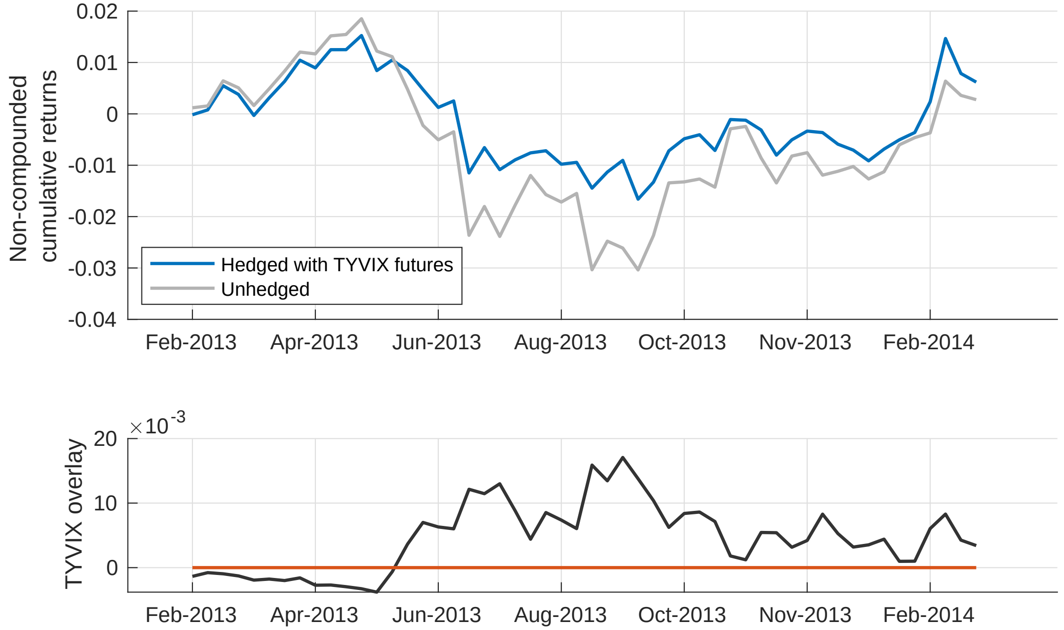
Backwardation premiums



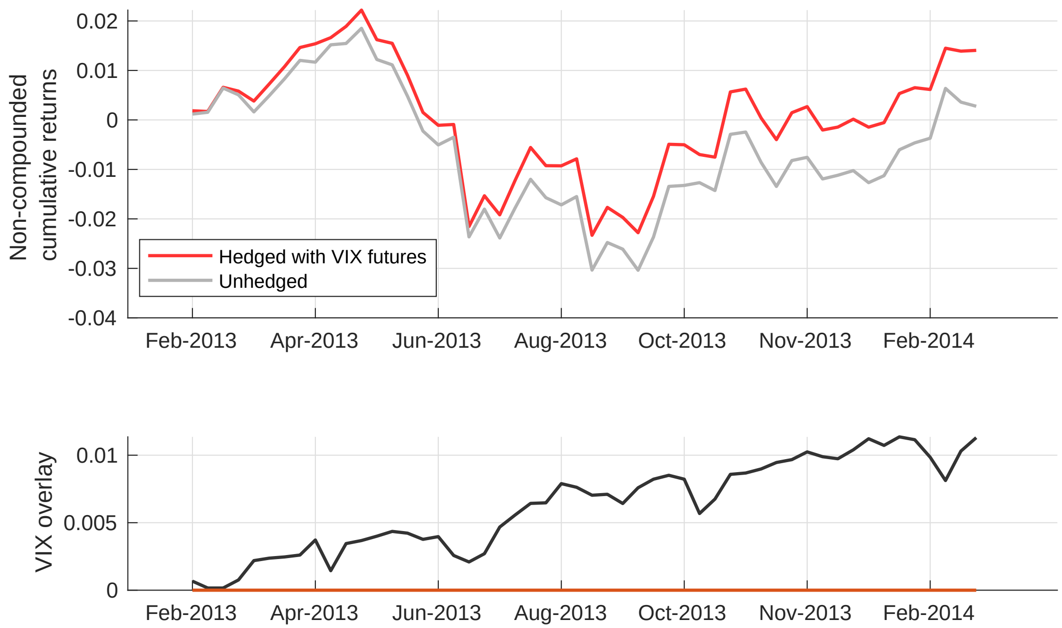

TYVIX index
Near-and-next-month hedge
Near-month hedge
Non-compounded cumulative returns (%)
600
500
400
300
200
100
0
-100
2006
2008
2010
2012
2014
2016

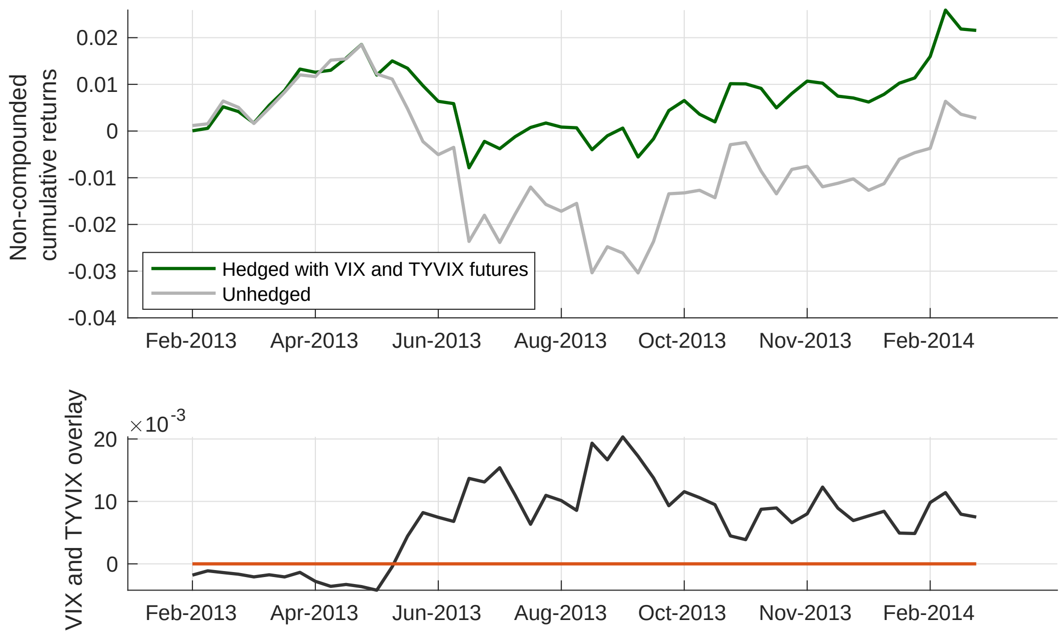
Overlaying fixed income portfolios





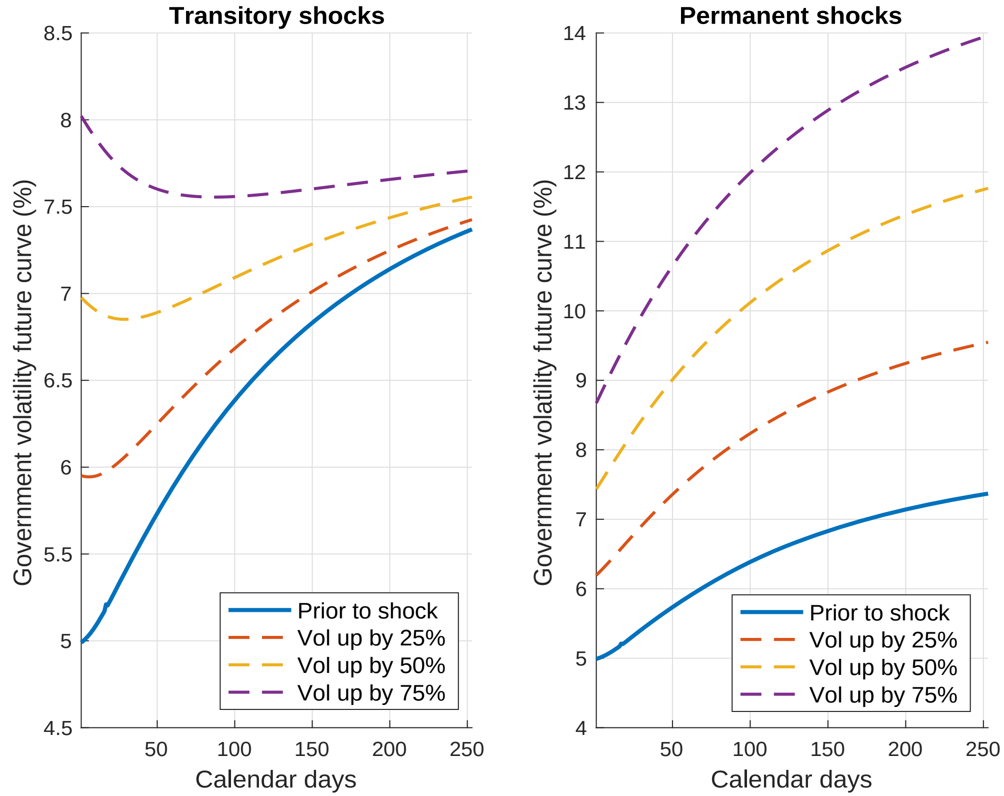
Non-compounded cumulative returns
0.02
0.01
0
-0.01
-0.02
-0.03
-0.04
Hedged with TYVIX futures
Unhedged
Feb-2013
Apr-2013
Jun-2013
Aug-2013
Oct-2013
Nov-2013
Feb-2014
TYVIX overlay
×10-3
20
10
0
Feb-2013
Apr-2013
Jun-2013
Aug-2013
Oct-2013
Nov-2013
Feb-2014


Non-compounded cumulative returns
0.02
0.01
0
-0.01
-0.02
-0.03
-0.04
Hedged with VIX futures
Unhedged
Feb-2013
Apr-2013
Jun-2013
Aug-2013
Oct-2013
Nov-2013
Feb-2014
VIX overlay
0.01
0.005
0
Feb-2013
Apr-2013
Jun-2013
Aug-2013
Oct-2013
Nov-2013
Feb-2014

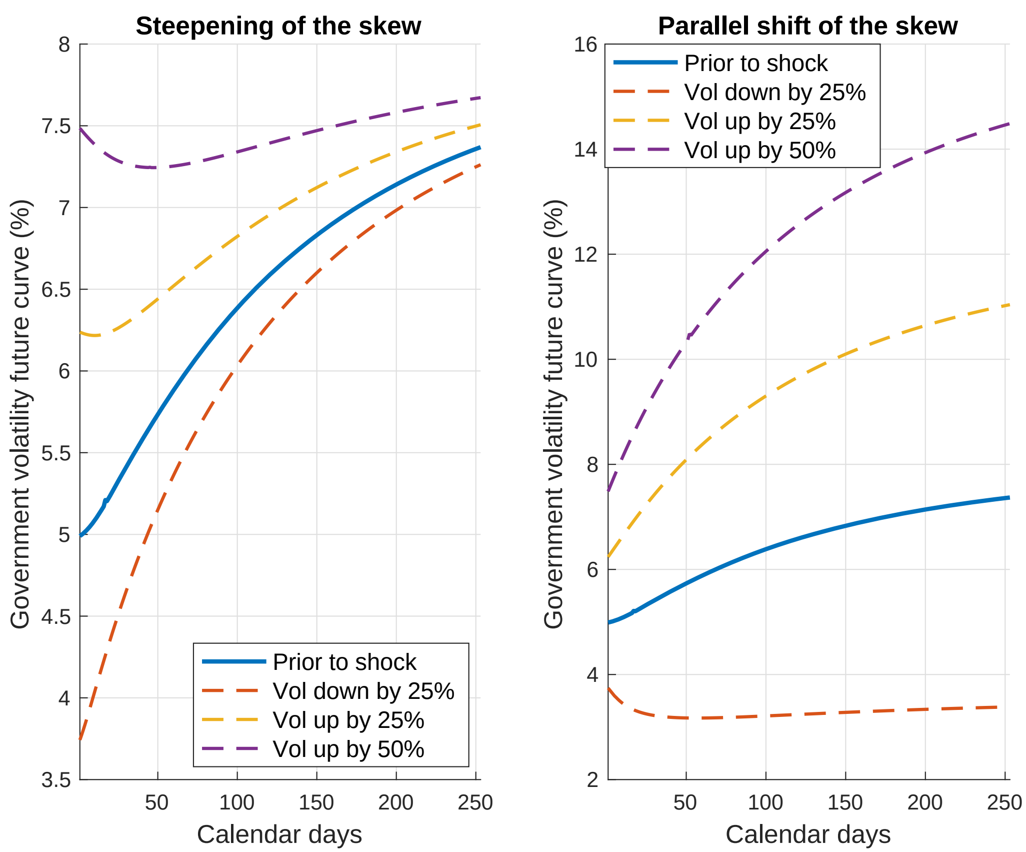

Non-compounded cumulative returns
0.02
0.01
0
-0.01
-0.02
-0.03
-0.04
Hedged with VIX and TYVIX futures
Unhedged
Feb-2013
Apr-2013
Jun-2013
Aug-2013
Oct-2013
Nov-2013
Feb-2014
VIX and TYVIX overlay
×10-3
20
10
0
Feb-2013
Apr-2013
Jun-2013
Aug-2013
Oct-2013
Nov-2013
Feb-2014

Scenarios

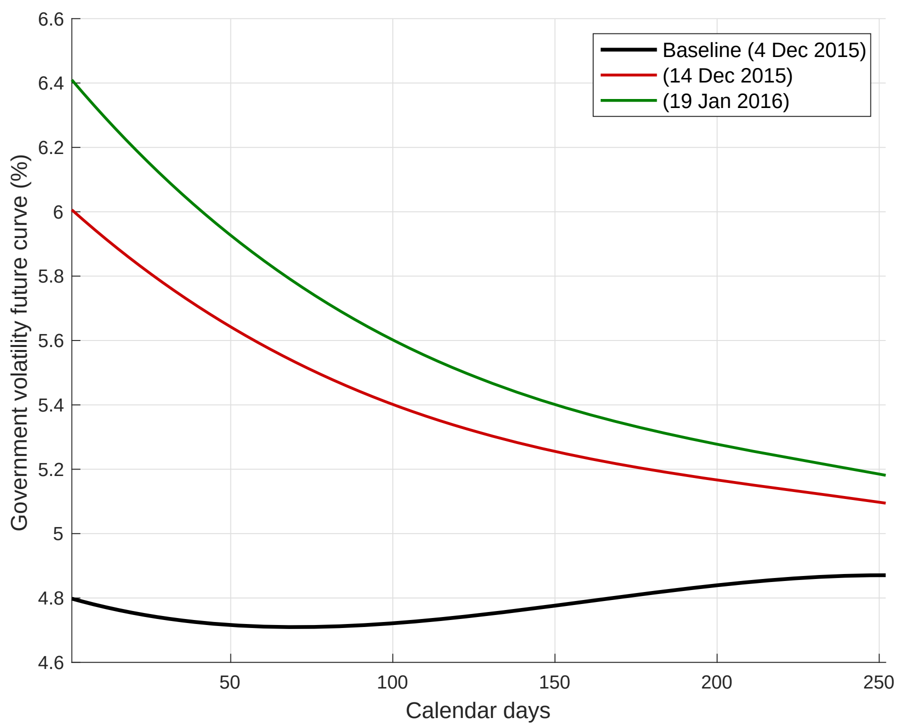
Transitory & permanent shocks



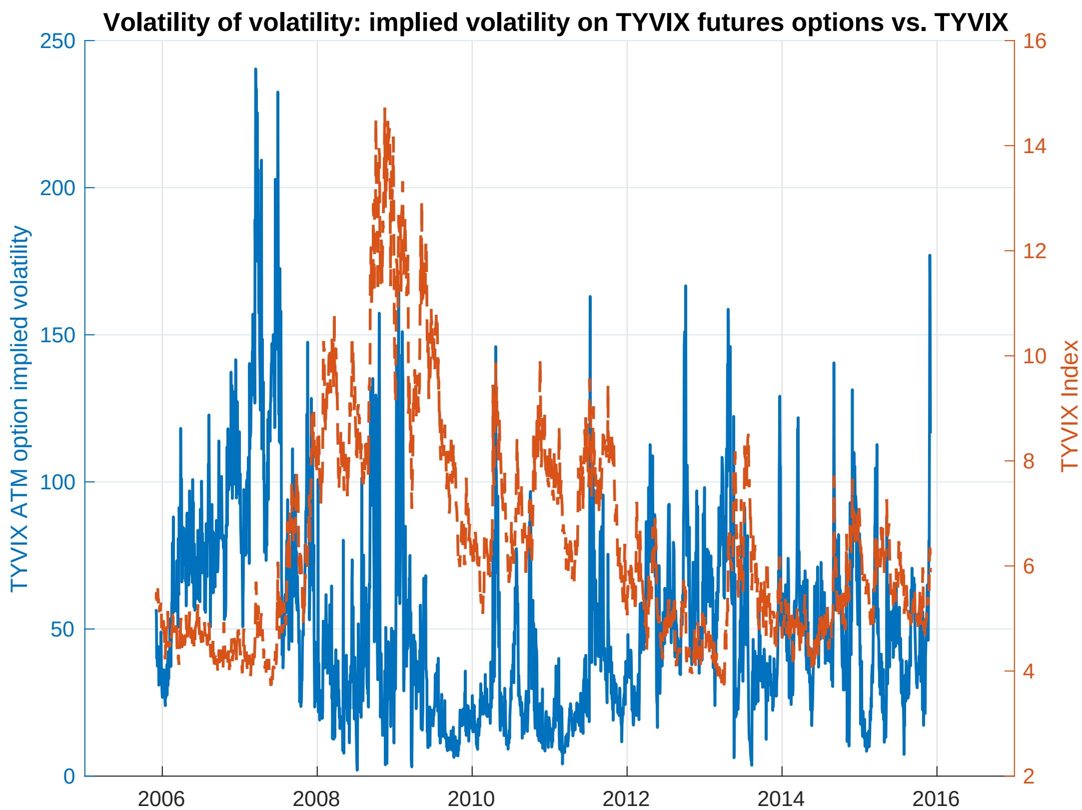
Shocks to the option skew



FED hike (Dec 2015)...
... and emerging markets turmoil (Jan 2016)



Volatility of volatility



Conclusion

- Fixed income vol: focus of renewed interest in recent literature
- Construct model to evaluate hedging instruments referenced to market-adjusted expectations of volatility in government bond markets
- Predictions:

- Predictions:
  - Futures markets on U.S. Treasury expected volatility are likely to experience frequent oscillations between backwardation and contango
    - in stark contrast to dynamics known in the equity space
    - hedging fixed income portfolios with gvt vol futures leads to mitigate drawdowns while avoiding large insurance costs
  - Model also used to ask whether derivatives on expected government volatility may help summarize or anticipate short-term or long-term uncertainty in monetary policy and other macroeconomic developments
    - shocks on the government vol curve tend to affect the short-end of the curve more than the long-term
  - Volatility of volatility
    - i.e., the implied volatility of hypothetical options referenced to government bond volatility futures
    - useful as a new instrument to monitor adverse market movements